Physics of Racing, Part 21:

The Magic Formula: Longitudinal Version

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Driving a car is a classic problem in ***control***. Here, we mean *control* in the technical sense of *control theory*, an established branch of engineering science (once again, I find www.Britannica.com to have a very nice, brush-up article on that term). In a more-or-less continuous fashion, the driver compares desired direction, speed, and acceleration with actual direction, speed, and acceleration. The driver uses visual input to sense actual direction and speed; and uses visceral, inertial feedback—the butt sensor—for actual acceleration. When the actual differs too much from the desired, the driver applies throttle, brake, steering, and gear selection to change the actual. These inputs cause the tires to react with the ground, which pushes back against the tires, and through the suspension, pushes the body of the car and driver. Drivers in high-speed circumstances can also generate desired aerodynamic forces, as in slipstreaming, in the "slingshot pass," and in the Earnhardt TIP maneuver, where the driver "takes the air off" the spoiler of the car in front of him.

Tires generate forces by sticking and sliding and everything in between. They transmit these forces to the wheels by elastic deformation. The elastic deformation is extremely complex and theoretical computation requires numerical solution of finite-element equations. However, despite fierce trade secrecy, industry and academia have reached apparent consensus in recent years on a formula that summarizes experimental and theoretical data. This so-called ***magic formula*** is not a solution to equations of motion—a solution in such a form is not feasible. It's just a convenient fitting of commonplace mathematical functions to data. It allows one to compute forces at a higher precision than something like RARS (see parts 16 and 19 of the Physics of Racing [*PhOR*]), but without integrating equations. Therefore, forces can be computed within a reasonable time, say in a real-time simulation program.

To understand the magic formula, we need first to define its inputs, which include ***slip***. Slip is an indirect measure of the fraction of the contact patch that is sticking. It is frequently asserted in the literature that a tire with no slip at all cannot create forces. It has taken me a very long time to accept this assertion. Why can I steer a tin-toy car with metal tires on a hard surface like Formica? If there is any slip in such tires, it is microscopic, yet there are sufficient forces to brake and steer, even if just a little. I finally caved in when I realized that the forces are minute, also. If there is any friction between the tire and the surface, there MUST be slip, as it is defined below. Though to a very small degree, the Formica and the tiny contact patches of the tin tires actually twist and stretch each other. The only way to eliminate slip completely is to eliminate GRIP completely. Any grip, and you will have slip.

There are two, slightly different flavors of the magic formula. The ***longitudinal*** one is the subject of this entire installment of PhOR, and we cover the ***lateral*** one in the next installment. Longitudinal slip is along the mean plane of the wheel and might also be called *circumferential* or *tangential.* It creates braking and accelerating forces. Lateral slip is our old friend grip angle [PhOR-10], and it generates cornering forces.

We write longitudinal slip as σ. It's defined by the *actual* angular velocity, ω, of a wheel plus tire under braking or acceleration, compared to the corresponding angular velocity of the same wheel plus tire when rolling freely. We write the free-rolling angular velocity as $\omega_0 = V/R_e$, where V is the current, instantaneous velocity of the hub centerline of the wheel with respect to the ground, and R_e is the ***effective radius***, a constant defined below. Since the dimensions of V are length/time, and the dimensions of any radius are length, the ratio, $\omega_0 = V/R_e$, has dimensions of inverse time. In fact, it should be viewed as measuring radians per unit time, radians being the natural, dimensionless measure of angular rotation. There are 2π radians in one rotation or one circumference of a circle, just as the length of the circumference is 2π times the radius.

Let's begin the discussion of longitudinal slip with a question. Consider a wheel-tire combination with 13-inch radius or 26-inch diameter, say a 255-50/16 tire on a 16-inch rim. The "50" in the tire specification is the ratio of the sidewall height to the tread width, which is also written into the specification as 255, millimeters understood. We get a sidewall height of 50 percent of 255 mm, which is 5.02 inch. Therefore, the total, ***unloaded radius***, half of the tread-to-tread diameter, is about 5+16/2 = 13 Inch. Now consider a rigid tire of the same radius, made, say, of steel or of wood with an iron tread like old Western wagon wheels. The question is whether, given a certain constant hub velocity, pneumatic tires spin faster than, slower than, or at the same speed as equivalent rigid tires?

At first glance, one might say, "Well, faster, obviously. Since the pneumatic tire compresses radially under the weight of the car, its radius is actually smaller than the unloaded radius at the point of contact, where it sticks and acquires linear velocity equal in magnitude and opposite in direction to the hub velocity. Since smaller wheels spin faster than larger ones at the same speed, the pneumatic tire spins faster than the equivalent rigid tire of the same unloaded radius. Let the unloaded, natural radius of the pneumatic tire be R, also the radius of the equivalent solid tire. If the hub has velocity V, the solid tire spins with angular velocity $\omega = V/R$. Since the ***loaded radius***, of the pneumatic tire, R_l, is smaller than R, V/R_l, the angular velocity of the loaded pneumatic tire, must be larger than V/R."

This is partly correct. The pneumatic tire-wheel combination *does* spin faster than a rigid wheel of the same unloaded radius, but it does *not* spin as fast as a rigid wheel of the same *loaded* radius, which is the height of the hub center off the ground under load. The reason is that the tire also compresses *circumferentially* or *tangentially*, setting up complex longitudinal twisting in the sidewall. The tangential speed of a particle of tread varies as the particle goes around the circumference of the tire.

Let's mentally follow a piece of tread around as the *wheel*, not necessarily the tire, turns at a constant radial velocity, ω_0. Imagine a plug of yellow rubber embedded in the tread, so that you could visually track it or photograph it with a movie camera or strobe system as it moves around the circumference. The rubber of the tread does not travel at constant speed, even though the wheel supporting the tire does. At the top of the tire, the radius is almost exactly R, the unloaded radius, so the tread moves with tangential velocity $R\omega_0$. As the yellow plug rolls around and approaches the contact patch from the front, it slows down in the bunched up area at the *leading edge* of the contact patch—just forward of it. There *is* a bunched-up area, because the tire is made up of elastic material that gets squeezed and stretched out of the contact patch and piles up ahead of the contact patch as it rolls into it from the direction of the leading edge. Eventually, the plug enters the patch, in the center of which it must move at speed $R_l\omega_0$ relative to the hub center, that is, backwards at a speed dictated by the *loaded* radius and the wheel velocity. We've assumed that the plug is not slipping on the ground at the point where it has speed $R_l\omega_0$ with respect to the hub. This means that it has speed zero with respect to the ground at that point.

The average of the tangential velocities around the wheel defines the effective radius, R_e, as follows. Let θ measure the angular position, from 0 to 2π, around the wheel. Suppose we knew the tangential velocity with respect to the hub center, $V(\theta)$, at every θ. We could easily measure this with our strobe light and cameras. $V(\theta)$ gives us the radius at every angular position via the equation $V(\theta)/\omega_0 = R(\theta)$, where ω_0 is the constant angular velocity of the wheel. The average would be computed by the following integral:

$$R_e = \frac{1}{2\pi}\int_0^{2\pi} R(\theta)\,d\theta = \frac{1}{2\pi\omega_0}\int_0^{2\pi} V(\theta)\,d\theta$$

Let's run some numbers. 10 mph is $14\frac{2}{3}$ feet/second or 176 inches/second. With an *unloaded* circumference of 26π inch/revolution, we get $176/26\pi = 2.154$ revs per second, or 129 RPM for each 10 MPH. Under ordinary circumstances, the effective radius will be no more than a few percent less than then the unloaded radius, and the RPMs should be, then, a few percent more than 129 RPM per 10 MPH. At 100 MPH, the tire is under considerable stress and spins at something over 1,300 RPM.

Now we're in a position to define longitudinal slip, written σ. We want a quantity that vanishes when the wheel rolls at constant speed, increases when the wheel accelerates the car by pulling the contact patch backwards, and decreases below zero when then wheel brakes the car by pushing the contact patch forward. Under acceleration, the wheel and tire combination will tend to spin a little faster than it would do while free rolling. We already know that, for a given V, the free-rolling angular velocity is $\omega_0 = V/R_e$, by definition. The *actual* angular velocity, ω, then, is higher under acceleration. So, if we know V, ω, and the constant R_e, then we can define the longitudinal slip as the ratio, minus 1, so that it's zero under free-rolling conditions:

$$\sigma = \frac{\omega}{\omega_0} - 1 = \frac{\omega}{V/R_e} - 1 = \frac{\omega R_e - V}{V}$$

Just looking at this formula, a free-rolling wheel has $\omega = \omega_0$, $\sigma = 0$, a locked-up wheel under braking has $\omega = 0$, $\sigma = -1$, and an accelerating wheel has a positive σ of any value.

The magic formula yields the longitudinal force, in Newtons, given some constants and dynamic inputs. The formula takes eleven empirical numbers that characterize a particular tire $\{b_0, b_1, \cdots, b_{10}\}$. The dynamic parameters are , F_z, or *weight*, in **KiloNewtons** on the tire, and the instantaneous slip, σ. The eleven numbers are measured for each tire. We borrow an example from *Motor Vehicle Dynamics* by Giancarlo Genta. On page 528, he offers the following numbers for a car that appears to be a Ferrari 308 or 328, to which I have added dimensions:

b_0	1.65	dimensionless
b_1	0	1/MegaNewton
b_2	1688	1/Kilo
b_3	0	1/MegaNewton
b_4	229	1/Kilo
b_5	0	1/KiloNewton

b_6	0	$1/(\text{KiloNewton})^2$
b_7	0	1/KiloNewton
b_8	-10	dimensionless
b_9	0	1/KiloNewton
b_{10}	0	dimensionless

Though the majority of these values are zero for the tires on this car, it is by no means always the case. In fact, the 'large-saloon' example just before the (alleged) Ferrari in Genta's book has *no* zeros.

We build up the magic formula in stages. The first helper quantity is $\mu_p = b_1 F_z + b_2$. This is an estimate of the ***peak, longitudinal coefficient of friction***, fitted as a linear function of weight (see Part 7 of PhORs). From this definition, we begin to see what's going with the dimensions. A typical, streetable sports car might weigh in at 3,000 lbs, which is about 3,000/2.2 = 1,500 * 0.9 = 1,350 kg, which is about 1,350 * 9.8 = 13,200 Newtons, or 13.2 KiloNewtons (look, ma, no calculator!). Let's assume each tire gets a quarter of that to start off with, or 3.3 KN. b_1 multiplies that number to give us something with dimensions of KiloNewton/MegaNewton, which we write simply as 1/Kilo (inventing units on-the-fly, one Mega = 1 Kilo squared). b_2 has the same dimensions, so it's kosher to add it in, yielding $\mu_p = 1688/\text{Kilo}$ in this case. The next step is the helper $D = \mu_p F_z$, which will be in Newtons. We now see the reason for the 1/Kilo unit. In our case, we get about D=(1700-12)*3.3=5610-40=5570 N. The important point is that D **is linear in** F_z, so μ_p acts, mathematically, like a coefficient of friction, as promised. b_2 is a pretty direct measurement of stickiness, times 1,000 for convenience. This model tire has a coefficient of friction of almost 1.7! Not my data, man.

The next step is to compute the product of a new helper, B, times b_0 and the aforecomputed D. The magicians who created the formula tell us that $Bb_0D = \left(b_3 F_z^2 + b_4 F_z\right)\exp\left(-b_5 F_z\right)$. This slurps up a few more of the magical eleven empirical numbers, and a pattern emerges. These b_i numbers serve as coefficients in polynomial expressions over F_z. So, $b_5 F_z$ is dimensionless, as must be the argument of the exponential function. $b_3 F_z^2 + b_4 F_z$ has dimensions of Newtons, as does the entire product. Therefore, B must be dimensionless. We need B in the next step, so let's solve for it now:

$$B = \frac{[Bb_0 D]}{b_0 D} = \frac{[Bb_0 D]}{b_0 \mu_p F_z} = \frac{(b_3 F_z + b_4)\exp(-b_5 F_z)}{\left[b_0 \mu_p = b_0 (b_1 F_z + b_2)\right]},$$

Where we've been able symbolically to divide out one factor of F_z, convenient especially for numerical computation, where overflow is an ever-present hazard. Continuing with our numerical sample, $b_3 F_z + b_4 = 229/\text{Kilo}$, the exponential is unity, and the numerator is

$$(b_0 = 1.65) * \left(\mu_p = \frac{1688}{\text{Kilo}}\right) \cong \frac{1688 + 844 + 169 + 85}{\text{Kilo}} = \frac{2786}{\text{Kilo}}$$

yielding $B = 229/2786 = 0.0822$. Most importantly, B **depends only weakly on** F_z. In the sample case, not at all, because $b_1 = b_3 = b_5 = 0$, but there are lots of other ways to characterize the algebraic dependence of B on F_z.

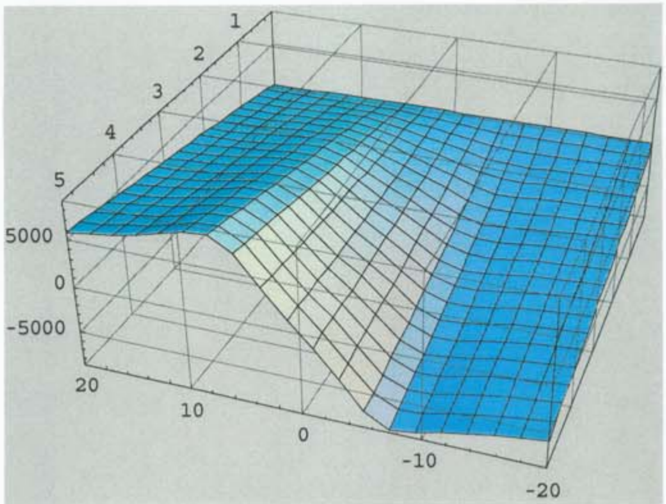
The next step is to account for the longitudinal slip with another helper, $S = (100\ \sigma + b_9 F_z + b_{10})$; in our sample case, this reduces to just $S = 100\ \sigma$.

Only one more helper is needed, and that's $E = (b_6 F_z^2 + b_7 F_z + b_8)$, very straightforward. The final formula is

$$F_x = D \sin\left(b_0 \tan^{-1}\left\{SB + E\left[\tan^{-1}(SB) - SB\right]\right\}\right)$$

Once again, don't try to find any physics in here: it's just a convenient formula that fits the data reasonably well. Plugging in numbers for $\sigma = 0$, because that's an easy sanity check to do in our heads, we see immediately the result is zero. Let's try $S = 10$, ten percent slip. $SB = 0.822$, $\tan^{-1}(0.822) = 0.688$, $E = -10$, so the argument of the outer arctangent is $SB - 10*(-0.266) = SB + 2.66 = 3.48$, $\tan^{-1}(3.48) = 1.29$, $1.29\, b_0 = 2.13$, $\sin(2.13) = 0.848$, and, finally, $D * 0.848 = 4720$ Newtons. Lots of longitudinal force for a 3,300 N vertical load!

Let's plot the whole formula:



The horizontal axis measures $S = 100\ \sigma$, which is really just slip in percent. The deep axis, going into the page, measures F_z from 5 KN, nearest us, to zero in the back. The vertical axis measures the result of applying the formula to our model tire, so it's longitudinal force—force of launching or braking. Notice that for a load of 5 KN, the model tire can generate almost 8 KN of force. Very sticky tire, as we've already noticed! Also notice that the generated force peaks at around $\sigma = 0.08$, or 8 percent. The peak would be something one could definitely feel in the driver's seat. Overcooking the throttle or brakes would produce a palpable reduction in g-forces as the tires start letting go. Worse than that, increasing braking or throttle beyond the peak leads to reduced grip. This is an instability area, where *increasing slip leads to decreasing grip.*

Finally, note that the function behaves roughly linearly with F_z, showing that it acts like a Newtonian coefficient of friction, albeit a different one for each value of slip.